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On a non-linear theory of thin jets. Part 2. A linear theory for small injection angles

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A non-linear potential problem which describes the injection of a high speed jet into a uniform flow of lower total head is linearized when the jet injection angle is small. The resulting linear problem is solved using the Wiener–Hopf technique. Numerical results for the position of the streamline separating the jet from the free stream and the pressure coefficient along the upstream wall are obtained for various injection angles.

1. Introduction

In two recent papers Ackerberg & Pal (1968) and Ackerberg (1968) [hereafter referred to as A–P and I, respectively] studied the interaction of a jet with a uniform flow for cases where the ratio of the free stream to jet dynamic heads, $\mu = \rho_2 q_{\infty 2}^2/\rho_1 q_{\infty 1}^2$, is small compared to unity (see figure 1). Here $\rho_{1,2}$ and $q_{\infty 1,2}$ are the fluid densities and speeds at an infinite distance downstream, and the subscripts 1 and 2 refer to the jet and the free stream, respectively. In this paper it is shown that when the jet injection angle α, measured between the initial jet direction and the downstream uniform flow direction, is small compared to unity, the non-linear boundary condition along the vortex sheet can be linearized, and only a single linear potential problem need be solved for $\alpha \ll 1$. A physical interpretation of this problem is given, and a solution is found using the Wiener–Hopf technique. Numerical results are obtained for the position of the bounding streamline separating the jet from the free stream and for the pressure coefficient along the upstream wall. Although the theory is not expected to be valid for large injection angles, a comparison is made with the numerical results obtained by A–P for normal injection and the agreement is fairly good.

2. Formulation

Introduce a co-ordinate system $z = x + iy$ with origin at O (see figure 1). In the external flow outside of the jet define the complex velocity potential $w^*(z) = \Phi + i\Psi$ and the complex velocity $dw^*/dz = u - iv = qe^{-i\theta}$. Each function will be analytic in z, and thus the logarithm of the non-dimensional complex velocity

$$\tilde{\Gamma}(w^*) \equiv \tilde{Q}(\Phi, \Psi) - i\tilde{\theta}(\Phi, \Psi) = \ln(q_{\infty 2}^{-1} dw^*/dz), \tag{2.1}$$

will be an analytic function of w^*. The fluid deflexion $\tilde{\theta}$ is related to the logarithm of the non-dimensional speed $\tilde{Q}$ by the Cauchy–Riemann equations.

Introduce the non-dimensional variables

$$Z = X + iY = \mu z/d \quad \text{and} \quad w = \phi + i\psi = (\mu/dq_{\infty 2})w^*,\dagger \tag{2.2}$$

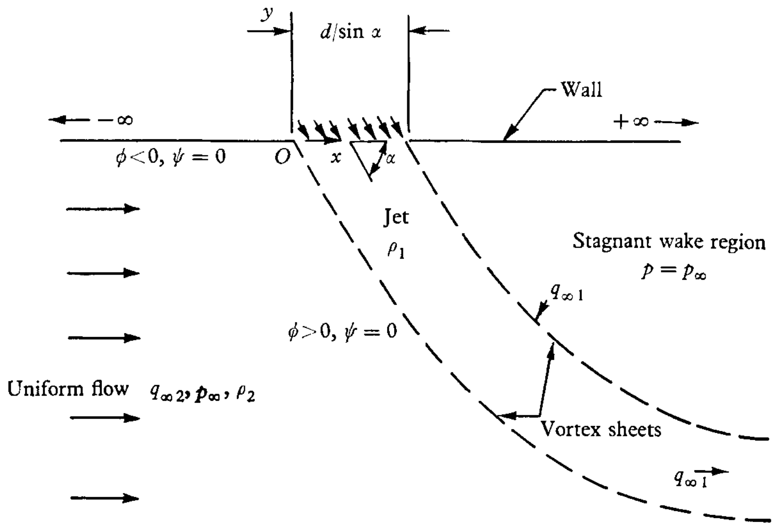

FIGURE 1. Region of flow in physical plane.

where d is the jet width. According to A–P or I, $\tilde{Q}(\Phi, \Psi)$ satisfies the following non-linear potential problem when $\mu \to 0$:

$$\nabla^2 \tilde{Q} = 0 \quad \text{for} \quad \psi \leqslant 0 \quad (-\infty \leqslant \phi \leqslant \infty), \tag{2.3}$$

$$\partial \tilde{Q}/\partial \psi = 0 \quad \text{for} \quad \psi = 0 \quad (\phi < 0), \tag{2.4}$$

$$\partial \tilde{Q}/\partial \psi = -\sinh \tilde{Q} \quad \text{for} \quad \psi = 0 \quad (\phi > 0), \tag{2.5}$$

$$\tilde{Q} \sim (\alpha/\pi) \ln |w| \quad \text{for} \quad |w| \to 0, \tag{2.6}$$

and

$$\tilde{Q} \to 0 \quad \text{for} \quad |w| \to \infty. \tag{2.7}$$

Here ∇^2 is the Laplacian operator with respect to (ϕ, ψ).

The linearization procedure

To obtain an expansion which is valid for small α, assume

$$\tilde{Q}(\phi, \psi) \sim \alpha Q(\phi, \psi) + o(\alpha).\ddagger \tag{2.8}$$

Substituting (2.8) into (2.3)–(2.7), expanding for $\alpha \to 0$ [assuming Q remains of $O(1)$], and equating the coefficients of α to zero, we obtain (2.3), (2.4), (2.6) and (2.7) with $\tilde{Q}$ replaced by Q and $\alpha = 1$.§ The main simplification occurs in (2.5),

† Note the slight change in notation from I where $\tilde{Q}$ was the leading term in an asymptotic expansion for $\mu \to 0$. To first order, $\tilde{Q}$ in I is equal to $\tilde{Q}$ in (2.1). Also, $\tilde{w}$ in I is defined here as w by (2.2).

‡ The next non-trivial term will be of $O(\alpha^3)$. Only the first-order solution will be considered here.

§ Hereafter when referring to these equations, we will assume that these substitutions have been made.

which is now replaced by the linear boundary condition

$$\partial Q/\partial \psi = -Q \quad \text{for} \quad \psi = 0 \quad (\phi > 0). \tag{2.9}$$

It is expected that this equation will be in error when

$$|w| = O(e^{-\pi/\alpha}), \tag{2.10}$$

due to the singularity condition (2.6). The linearization changes the algebraic singularity of $\partial \tilde{Q}/\partial \psi$ on $\psi = 0$ when $\phi \to 0+$ [which should be of the $O(\phi^{-\alpha/\pi})$] into a logarithmic singularity. Nevertheless, we will find *a posteriori* that this modification only affects the streamline curvature near the jet exit and not the values of $\tilde{Q}$ and $\tilde{\theta}$, both of which may be determined correctly to $O(\alpha)$ everywhere.†

A physical interpretation of the potential problem

The potential problem defined by (2.3), (2.4), (2.6), (2.7) and (2.9) has a simple physical interpretation in terms of a second unrelated flow. Let us suppose that

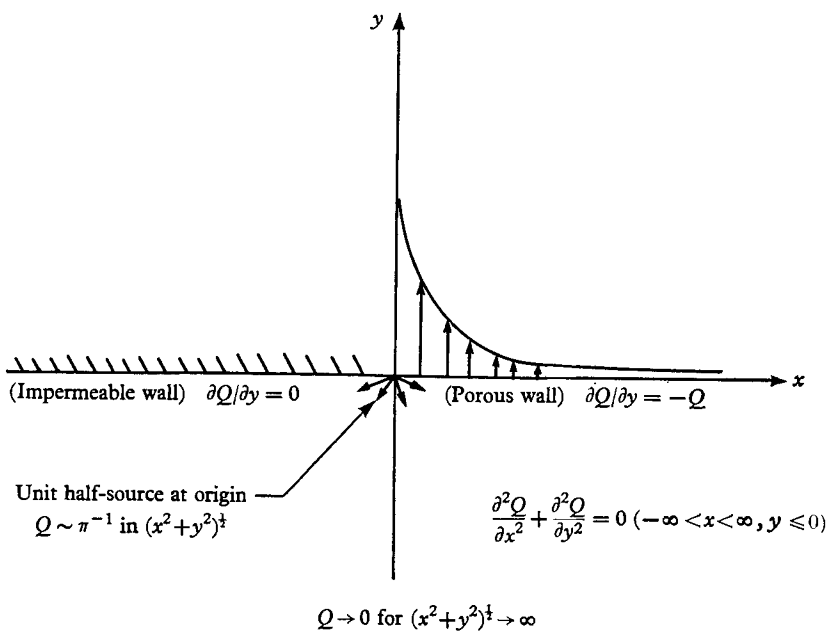

FIGURE 2. Physical interpretation of the linear potential problem.

Q represents a velocity potential, and we interpret (ϕ, ψ) as the physical coordinates (x, y). This new flow takes place in the lower half-plane $y < 0$ with a

† It has been pointed out in I that in the non-linear theory the curvature determined from $\tilde{Q}$ and $\tilde{\theta}$ in the region near the jet exit [$\phi = O(\mu^{\pi/(\pi-\alpha)})$] is in error by $O(\mu)$ for $\alpha \neq \frac{1}{2}\pi$. When $\alpha \to 0$, the region given by (2.10) will be much smaller than the aforementioned region provided $\alpha/\pi \ll |\ln \mu|^{-1}$. Since this condition will usually be met in practical situations, there is no point in correcting the curvature to $O(\alpha)$. It is worthwhile noting that with the linearization the curvature of the bounding streamline at the jet exit is zero; this agrees with the correct value given by the inner solution when $\alpha < \frac{1}{2}\pi$ (see I). Therefore, the bounding streamlines determined here for $\alpha \to 0$ may be more realistic than those determined from the non-linear equations.

half-source of strength one placed at the origin (see figure 2). The boundary conditions imply that the wall $y = 0$, $x < 0$ is impermeable to fluid whereas fluid is being withdrawn through $y = 0$, $x > 0$ at a rate proportional to the negative of the local value of the velocity potential. From (2.7) we must have no net sink (or source) strength at infinity. This can only be the case if precisely one unit of fluid per unit time is being withdrawn along the entire porous wall. Thus, we require

$$\int_{0+}^{\infty} \left(\frac{\partial Q}{\partial \psi}\right)_{\psi=0} d\phi = -\int_{0+}^{\infty} Q(\phi, 0)\, d\phi = 1.\dagger \tag{2.11}$$

If this condition is not met, the jet will not be aligned with the uniform flow at an infinite distance downstream. This follows from the Cauchy–Riemann relation $\partial Q/\partial \psi = \partial \theta/\partial \phi$, relating Q with its conjugate harmonic function θ, and noting that $\theta(0+, 0) = -1$.

When the linear problem is considered in this way there can be little doubt that a solution exists. The uniqueness, on the other hand, can be established from a variational principle and will be discussed elsewhere.

3. The integral equation and its solution by the Wiener–Hopf technique

The solution of the potential problem can be found by first formulating it as an integral equation and then applying the Wiener–Hopf technique. To cast the problem into a convenient form, we consider the slightly more general equation

$$\nabla^2 Q - k^2 Q = 0, \tag{3.1}$$

where k is a positive parameter which will be allowed to approach zero at a convenient point in the analysis. The appropriate Green's function which has a zero normal derivative along $\psi = 0$ is

$$G(\phi, \psi | s, t) = -(2\pi)^{-1}[K_0(k[(\phi - s)^2 + (\psi - t)^2]^{\frac{1}{2}}) + K_0(k[(\phi - s)^2 + (\psi + t)^2]^{\frac{1}{2}})], \tag{3.2}$$

where K_0 is the modified Bessel function of the second kind of order zero. If we apply Green's theorem, and allow for the boundary conditions and the logarithmic singularity at the origin, the following integral equations can be obtained for Q:

$$Q(\phi, 0) = -\pi^{-1} \int_0^{\infty} Q(s, 0) K_0(k|\phi - s|)\, ds - \pi^{-1} K_0(k|\phi|) \quad \text{for} \quad -\infty < \phi < \infty. \tag{3.3}$$

Define the new functions

$$f(\phi) = \begin{cases} Q(\phi, 0) & \text{for} \quad \phi > 0, \\ 0 & \text{for} \quad \phi < 0, \end{cases} \tag{3.4}$$

and

$$g(\phi) = \begin{cases} 0 & \text{for} \quad \phi > 0, \\ Q(\phi, 0) & \text{for} \quad \phi < 0. \end{cases} \tag{3.5}$$

† This is equivalent to the condition that the integral of the normal derivative of a harmonic function around a closed boundary must vanish to ensure the absence of a net source or sink strength within the contour. Here we are insisting on having no singularities at infinity so that the integral around the infinite contour must vanish.

The integral equation may now be written

$$f(\phi)+g(\phi) = -\pi^{-1}\int_{-\infty}^{\infty} f(s)K_0(k|\phi-s|)\,ds-\pi^{-1}K_0(k|\phi|) \quad \text{for} \quad -\infty < \phi < \infty. \tag{3.6}$$

Introduce the Fourier transform and its inverse

$$F_-(\omega) = \int_{-\infty}^{\infty} e^{-i\phi\omega}f(\phi)\,d\phi, \quad f(\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\phi\omega}F_-(\omega)\,d\omega. \tag{3.7}$$

From the definition (3.4), $F_-(\omega)$ will be an analytic function in the lower half-plane $\operatorname{Im}\omega < 0$; this is denoted by the suffix $-$. Similarly $G_+(\omega)$, the Fourier transform of $g(\phi)$, will be an analytic function in the upper half-plane. On taking the Fourier transform of (3.6) we find

$$F_-(\omega)+G_+(\omega) = -\pi^{-1}\bar{K}(\omega)F_-(\omega)-\pi^{-1}\bar{K}(\omega), \tag{3.8}$$

where

$$\bar{K}(\omega) = \pi(k^2+\omega^2)^{-\frac{1}{2}}, \tag{3.9}$$

is the Fourier transform of $K_0(k|\phi|)$ and is a regular function in the ω-plane which has been cut along the imaginary axis from ik to $i\infty$ and $-i\infty$ to $-ik$. Rewriting (3.8) we obtain

$$F_-(\omega)[1+(k^2+\omega^2)^{-\frac{1}{2}}] = -G_+(\omega)-(k^2+\omega^2)^{-\frac{1}{2}}. \tag{3.10}$$

To apply the Wiener–Hopf technique it is necessary to write the coefficient of $F_-(\omega)$ as the ratio of two functions, the numerator being analytic in the lower half-plane $\operatorname{Im}\omega < k$, and the denominator being analytic in the upper half-plane $\operatorname{Im}\omega > -k$. Therefore, write

$$1+(k^2+\omega^2)^{-\frac{1}{2}} = L_-(\omega)/L_+(\omega), \tag{3.11}$$

where

$$\ln L_+(\omega) = (2\pi i)^{-1}\int_{-\infty+ic}^{\infty+ic} (\omega-\xi)^{-1}\ln[1+(\xi^2+k^2)^{-\frac{1}{2}}]\,d\xi, \tag{3.12}$$

$$\ln L_-(\omega) = (2\pi i)^{-1}\int_{-\infty+id}^{\infty+id} (\omega-\xi)^{-1}\ln[1+(\xi^2+k^2)^{-\frac{1}{2}}]\,d\xi, \tag{3.13}$$

and $-k < c < \operatorname{Im}\omega < d < k$.† To evaluate these integrals it is convenient first to differentiate with respect to ω and then displace the contours so that the integral for L_- (say) embraces the branch cut from $i\infty$ to ik. Taking into account the contribution from the neighbourhood of the branch point ik, we find after letting $k\to 0$

$$L_-'(\omega)/L_-(\omega) = -(2\omega)^{-1}+\frac{i}{\pi}\int_0^{\infty} [(\xi+i\omega)(1+\xi^2)]^{-1}\,d\xi.\ddagger \tag{3.14}$$

To complete the integrations, let ω be a point on the negative imaginary axis and put $\omega = -it$ with $t > 0$. We find

$$L_-(-it) = C(1+t^{-2})^{\frac{1}{4}}\exp\left\{-\pi^{-1}\int_0^t (1+s^2)^{-1}\ln s\,ds\right\}, \tag{3.15}$$

† The method of splitting and the Wiener–Hopf technique are fully discussed in a book by Noble (1958).

‡ Primes denote differentiation with respect to ω.

where C is a constant of integration.† To determine $L_-(\omega)$ elsewhere, introduce a cut along the positive imaginary axis. Thus, if ω is real and positive,

$$L_-(\omega) = Ce^{-\frac{1}{4}\pi i}(1+\omega^{-1})^{\frac{1}{2}} \exp\left\{-(i/\pi)\int_0^\omega (1-s^2)^{-1}\ln s\, ds\right\}. \tag{3.16}$$

Similarly, if $\omega = it$ with t real and positive,

$$L_+(it) = C(1+t^{-2})^{-\frac{1}{4}} \exp\left\{\pi^{-1}\int_0^t (1+s^2)^{-1}\ln s\, ds\right\}. \tag{3.17}$$

Here the constant of integration has been chosen so that (3.11) is satisfied when $k = 0$ and ω is real (see the footnote † below).

We now substitute (3.11) into (3.10) and rearrange to obtain

$$L_-(\omega)[1+F_-(\omega)] = L_+(\omega)[1-G_+(\omega)]. \tag{3.18}$$

If we assume, for the moment, that $k \neq 0$ and F_- and G_+ are analytic in the strip $|\mathrm{Im}\,\omega| < k$, the left-hand side of (3.18) defines a function which is analytic in the lower half-plane $\mathrm{Im}\,\omega < k$. Similarly the right-hand side is analytic in the upper half-plane $\mathrm{Im}\,\omega > -k$. It follows that both sides must equal a function which is analytic everywhere, except possibly at the point at infinity. The behaviour for $\omega \to \infty$ depends on the nature of $f(\phi)$ and $g(\phi)$ when $\phi \to 0$. To satisfy the singularity condition (2.6), we assume (and verify *a posteriori*) that the correct analytic function is a constant A. It follows then from (3.18) that

$$F_-(\omega) = -1 + A/L_-(\omega), \tag{3.19}$$

and a similar expression may be written for $G_+(\omega)$. Expanding (3.16) for $\omega \to \infty$,

$$L_-(\omega) \sim C[1-(i/\pi\omega)(1+\ln\omega)+(2\omega)^{-1}-2(2\pi\omega)^{-2}(\ln\omega)^2+\ldots]. \tag{3.20}$$

Substituting into (3.19), we obtain for $\omega \to \infty$

$$F_-(\omega) \sim -1+(A/C)[1+(i/\pi\omega)(1+\ln\omega)-(2\omega)^{-1}-2(2\pi\omega)^{-2}(\ln\omega)^2+\ldots]. \tag{3.21}$$

The correct logarithmic singularity will result at the origin by choosing $A/C = 1$. If each term of (3.21) is then interpreted separately, an asymptotic expansion for $\phi \to 0+$ can be found in the form‡

$$f(\phi) \sim \pi^{-1}\ln\phi + \pi^{-1}(\gamma-1) + (2\pi^2)^{-1}\phi(\ln\phi)^2 + O(\phi\ln\phi) \quad \text{for} \quad \phi \to 0+, \tag{3.22}$$

where Euler's constant $\gamma = 0{\cdot}5772\ldots$. The expansion for $g(\phi)$ is the same to this number of terms when $\phi \to 0-$ provided ϕ is replaced by $|\phi|$ in the arguments of the logarithms. A more general expansion for $\Gamma(w) = Q(\phi,\psi) - i\theta(\phi,\psi)$ when $w \to 0$ is

$$\Gamma(w) \sim \pi^{-1}\ln W + \pi^{-1}(\gamma-1) - W[(2\pi^2)^{-1}(\ln W)^2 + (\gamma-2)\pi^{-2}\ln W + O(1)] + \ldots, \tag{3.23}$$

where $W = e^{\pi i}w$ and $0 \geqslant \arg w \geqslant -\pi$.§

† Since we are determining the ratio L_-/L_+, C is completely arbitrary provided it appears as a multiplicative factor in L_+ and (3.11) is satisfied.

‡ A very convenient table for this purpose has been compiled by Geller (1963).

§ A comparison with the asymptotic expansion of $\tilde{\Gamma}(w)$ from the non-linear theory [see (4.14) of I] shows a difference in the analytical form of the third- and other higher-order terms. This accounts for the difference in the curvature of the bounding streamline at the jet exit.

To complete the determination of f and g we use the inversion integral (3.7) to obtain

$$f(\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}[-1+C/L_{-}(\omega)]\,e^{i\phi\omega}d\omega, \tag{3.24}$$

and

$$g(\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}[1-C/L_{+}(\omega)]\,e^{i\phi\omega}d\omega. \tag{3.25}$$

Both integrals may be evaluated in a similar way, which for the case of $f(\phi)$ involves deforming the contour so that it embraces both sides of the positive imaginary axis, and then using (3.11) and (3.17) to find the values of $L_{-}(\omega)$ on either side of the cut. The details are not very interesting and we simply give the results:

$$f(\phi) = -\pi^{-1}\int_{0}^{\infty} e^{-\phi t}t^{\frac{1}{2}}(1+t^2)^{-\frac{3}{4}}\exp\left\{-\pi^{-1}\int_{0}^{t}(1+s^2)^{-1}\ln s\,ds\right\}dt \quad \text{for} \quad \phi > 0, \tag{3.26}$$

and

$$g(\phi) = -\pi^{-1}\int_{0}^{\infty} e^{\phi t}t^{-\frac{1}{2}}(1+t^2)^{-\frac{1}{4}}\exp\left\{\pi^{-1}\int_{0}^{t}(1+s^2)^{-1}\ln s\,ds\right\}dt \quad \text{for} \quad \phi < 0. \tag{3.27}$$

Asymptotic expansions can be found for $|\phi|\to\infty$ by expanding the integrands of (3.26) and (3.27) for $t\to 0$ and integrating term-by-term. In this way we obtain [see Geller (1963)]

$$f(\phi) \sim -(2\sqrt{\pi})^{-1}\phi^{-\frac{3}{2}} - (\tfrac{3}{4}\pi^{-\frac{3}{2}})\phi^{-\frac{5}{2}}\ln\phi + \dots \quad \text{for} \quad \phi\to\infty, \tag{3.28}$$

and

$$g(\phi) \sim -(\pi|\phi|)^{-\frac{1}{2}} + (2\pi^{\frac{3}{2}})^{-1}|\phi|^{-\frac{3}{2}}\ln|\phi| + \dots \quad \text{for} \quad \phi\to-\infty. \tag{3.29}$$

The first few terms in these expansions have the same analytical form as those in the non-linear theory because, when $Q\to 0$, (2.5) is approximated by (2.9) to a high degree of accuracy.

Once $f(\phi)$ is known, the deflexion of the bounding streamline can be determined from the Cauchy–Riemann equations and (2.11) with the upper limit replaced by the variable limit ϕ, i.e.

$$\theta(\phi, 0) = -1 - \int_{0}^{\phi} f(t)\,dt \quad \text{for} \quad \phi > 0.\dagger \tag{3.30}$$

4. Calculation of the bounding streamline and the coefficient of the pressure

Numerical values for $f(\phi)$, $g(\phi)$ and $\theta(\phi > 0, 0)$, obtained on an IBM 7040 computer with Simpson's rule, are displayed in figures 3 and 4.‡ The details of

† It might be considered more accurate to determine $\theta(\phi, 0)$ from the non-linear relation (2.5). However, it is unlikely that

$$-\int_{0}^{\phi}\sinh[f(t)]\,dt = 1,$$

and as a result the jet will not be aligned with the uniform stream at an infinite distance downstream.

‡ Numerical tables of these functions have been lodged with the editor and are available on request.

the numerical calculations are lengthy and some of the more important aspects are discussed in the appendix.

The calculation of the position of the bounding streamline and the coefficient of the pressure along the upstream wall can be found using (2.1) and noting that to first order

$$\tilde{\Gamma}(w) = \alpha[Q(\phi,\psi)-i\theta(\phi,\psi)].$$

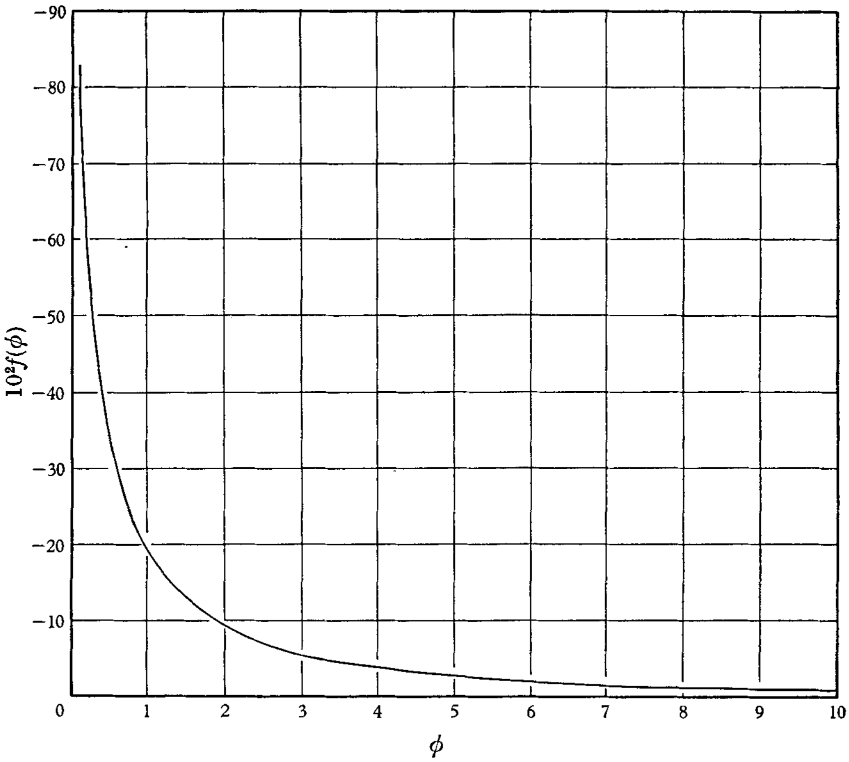

FIGURE 3. The function $f(\phi)$ versus ϕ.

Thus,

$$X = \int_0^\phi \exp[-\alpha f(t)]\cos[\alpha\theta(t,0)]\,dt \quad \text{on} \quad \psi = 0 \quad (\phi > 0), \tag{4.1}$$

$$Y = \int_0^\phi \exp[-\alpha f(t)]\sin[\alpha\theta(t,0)]\,dt \quad \text{on} \quad \psi = 0 \quad (\phi > 0), \tag{4.2}$$

and for $y = 0$, $x < 0$

$$C_p(X) = (p-p_\infty)/(\tfrac{1}{2}\rho_2 q_{\infty 2}^2) = 1-\exp[2\alpha g(\phi)] \quad \text{on} \quad \psi = 0 \quad (\phi < 0), \tag{4.3}$$

where now
$$X = \int_0^\phi \exp[-\alpha g(t)]\,dt \quad \text{on} \quad \psi = 0 \quad (\phi < 0), \tag{4.4}$$

and p is the static pressure.

Near the jet exit, (4.1) and (4.2) may be evaluated using the asymptotic formula (3.23). We find to first order after eliminating ϕ

$$Y \sim -X\tan\alpha + \ldots \quad (\alpha \neq \tfrac{1}{2}\pi) \quad \text{for} \quad X, Y \to 0 \quad \text{along} \quad \psi = 0 \quad (\phi > 0). \tag{4.5}$$

At large distances from the jet exit, the bounding streamline has the same asymptotic form (to the first few orders) as that found by A–P in the non-linear theory. Thus

$$X \sim (\pi/4\alpha^2)(Y+\alpha^2/2)^2 - (\alpha^2/2\pi)\ln[(\pi/4\alpha^2)(Y+\alpha^2/2)^2] + \text{const.} + o(1)$$
$$\text{for } X, Y \to \infty \quad \text{on} \quad \psi = 0 \quad (\phi > 0).\dagger \quad (4.6)$$

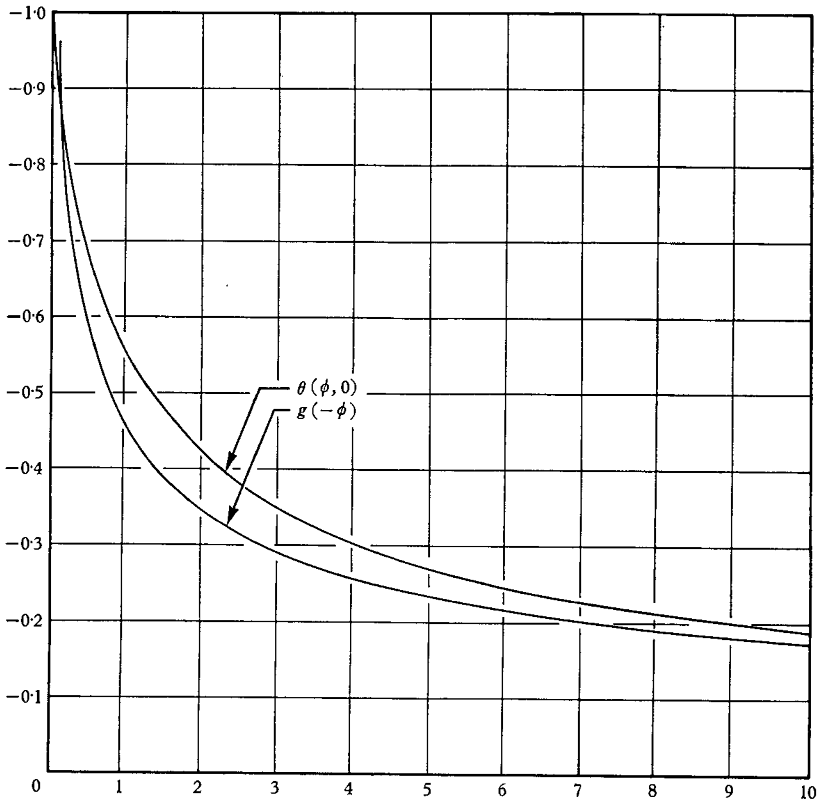

FIGURE 4. The functions $g(-\phi)$ and $\theta(\phi, 0)$ versus ϕ.

Numerical results

The shapes of the bounding streamlines for a number of injection angles are shown in figure 5. For small values of α (< 20°) the curves are very shallow and the theory is probably quite accurate near the jet exit where the spreading effect of viscosity will be small. Although the theory is not expected to be valid for large α, a comparison has been made with the numerical results for 90° injection obtained from the non-linear theory (see A–P). The linear theory gives a deeper jet penetration, and this could have been predicted from the physical interpretation in §2 by noting that, since $|\sinh Q| \geqslant |Q|$, more fluid will be withdrawn locally

† The value of A_0 in (4.8) of A–P can be given a definite value for the problem considered here because the constants in the asymptotic formulae (3.28) and (3.29) are known analyrically [cf. (4.4) of A–P]. Thus, $A_0 = -\alpha/\pi^{\frac{1}{2}}$.

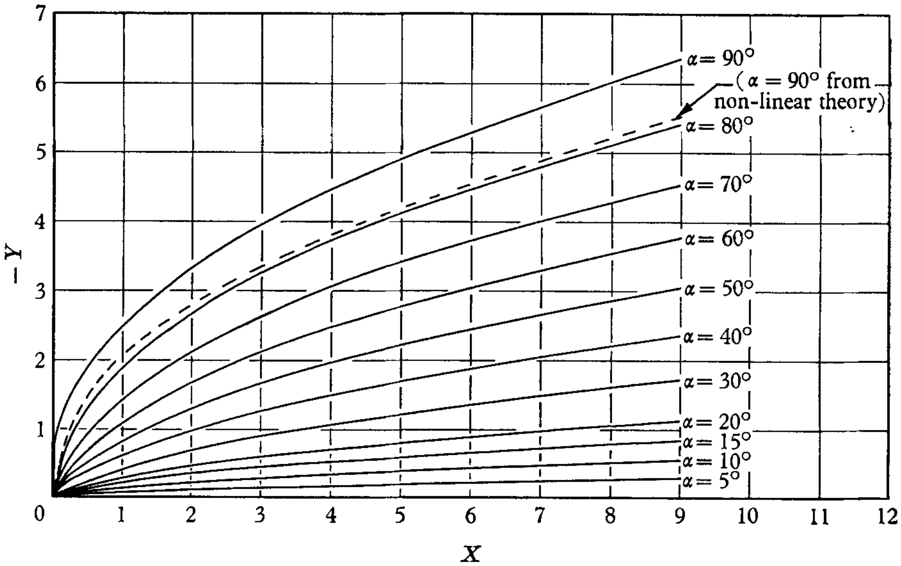

FIGURE 5. The bounding streamline for various values of α.

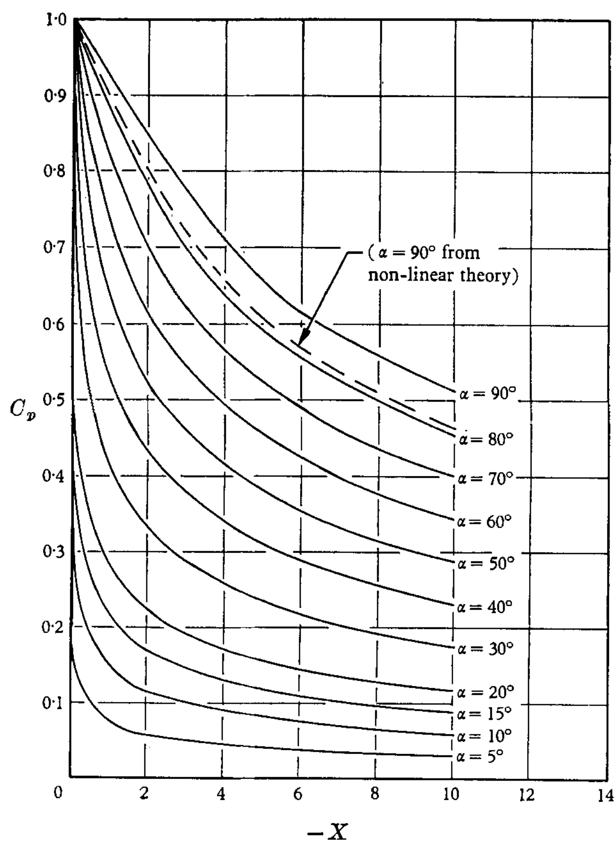

FIGURE 6. The coefficient of the pressure along the upstream wall for various values of α.

through the porous wall in the non-linear case and the jet penetration will be attenuated more rapidly.†

The coefficient of the pressure along the upstream wall is displayed for different injection angles in figure 6. It is interesting to note for small α the 'boundary layer' type of transition from the stagnation pressure at $X = 0-$ to almost free-stream static pressure within a very small distance from the jet exit [$X = O(2)$]. This result may partially account for the success of jet-flap theory in which the stagnation point is ignored [see Woods (1961, p. 409ff.)]. The difference in C_p for the linear and non-linear theory when $\alpha = 90°$ can also be explained by the argument given in the last paragraph.

It would be interesting and worthwhile to know if experimental results near the jet exit agree with the theory. An experimental apparatus with a liquid jet issuing into a uniform gas flow would have the advantage that μ could be made very small with moderate values of the velocities due to the density ratio. When $\alpha \leqslant \frac{1}{2}\pi$ the jet curvature is of $O(\mu)$ everywhere and surface tension effects, as well as viscous spreading, might not be too important close to the jet exit.

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Appendix

For numerical calculations it is convenient to write (3.26) and (3.27) as follows:

$$f(\phi) = -\pi^{-1}\int_0^1 e^{-\phi t} t^{\frac{1}{2}}(1+t^2)^{-\frac{3}{4}}[H(t)]^{-1}dt - \pi^{-1}E_1(\phi)$$
$$-\pi^{-1}\int_1^\infty e^{-\phi t}\{t^{\frac{1}{2}}(1+t^2)^{-\frac{3}{4}}[H(t)]^{-1} - t^{-1}\}dt \quad \text{for} \quad \phi \geqslant 0, \qquad (\text{A}\,1)$$

$$g(\phi) = -\pi^{-1}\int_0^1 t^{-\frac{1}{2}}\{e^{\phi t}(1+t^2)^{-\frac{1}{4}}H(t) - 1\}dt - \pi^{-1}[2 + E_1(-\phi)]$$
$$-\pi^{-1}\int_1^\infty e^{\phi t}\{t^{-\frac{1}{2}}(1+t^2)^{-\frac{1}{4}}H(t) - t^{-1}\}dt \quad \text{for} \quad \phi \leqslant 0, \qquad (\text{A}\,2)$$

where the exponential integral $E_1(\phi)$ is defined by

$$E_1(\phi) = \int_\phi^\infty t^{-1}e^{-t}dt, \qquad (\text{A}\,3)$$

and

$$H(t) = \exp\left\{\pi^{-1}\int_0^t (1+s^2)^{-1}\ln s\, ds\right\}. \qquad (\text{A}\,4)$$

In this form each integral is convergent for all ϕ for which it is defined, and the logarithmic singularity when $|\phi| \to 0$ is clearly displayed through $E_1(\phi)$.

† This assumes that the Q used in both boundary conditions is the same. For small values of α, Q will be uniformly valid to $O(1)$. Therefore, providing that integrals of higher-order terms do not contribute to the flux withdrawn through the wall (and there is no obvious reason why they should) it is probable that the bounding streamline from the linear theory represents an upper bound to the jet penetration.

If the infinite integrals in (A 1) and (A 2) are truncated at $t = T$, it is easy to show that the error will be of the form $DE_1(|\phi|T)$, where the constant $D < 5{\cdot}0$. Thus, to make the error of the $O(10^{-5})$ say, we should choose $T \approx 8/|\phi|$.

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